EE8725 Homework 5

Volker Landenberger

1 Optimization of a non-linear function with one inequality constraint

The function $f(x)$ with one non-linearity constraint $g(x)$ is to be minimized using the *Kuhn-Tucker-Condition* and the *Interior Point Method.*

$$\begin{array}{l} \min f(x) \\ s.t. \\ g(x) \leq 0 \end{array} \tag{1.1}$$

1.1 Kuhn-Tucker (KKT)

The Lagrangian is:

$$L(x, \mu) = f(x) + \mu g(x) \tag{1.2}$$

According to KKT, the necessary condition for a minimum at (x^*, μ^*) is:

$$\begin{array}{rcllcl} \nabla_x L(x^*, \mu^*) & = & \nabla f(x^*) + \mu^* \nabla g(x^*) & = & 0 \\ & & g(x^*) & \leq & 0 \\ & & \mu^* g(x^*) & = & 0 \\ & & \mu^* & \geq & 0 \end{array} \tag{1.3}$$

1.2 Interior Point Method with Logarithmic Barrier Function

Equation 1.1 can be rewritten by means of the slack variable s and the logarithmic barrier function $B(x) = -\mu^k ln(s)$ as follows:

$$\begin{array}{l} \min f(x) - \mu^k ln(s) \\ s.t. \\ g(x) + s = 0 \end{array} \tag{1.4}$$

That will force $s \geq 0$.

The Lagrangian of equation 1.4 yields:

$$L(x, s, \lambda) = f(x) - \mu^k \ln(s) + \lambda(g(x) + s) \tag{1.5}$$

and the minimum can be found at (x^*, λ^*, s^*) with the necessary condition:

$$\begin{array}{rcccl} \nabla_x L(x^*, s^*, \lambda^*) & = & \nabla f(x^*) + \lambda^* \nabla g(x^*) & = & 0 \\ \nabla_\lambda L(x^*, s^*, \lambda^*) & = & g(x^*) + s^* & = & 0 \\ \nabla_s L(x^*, s^*, \lambda^*) & = & \frac{\mu^k}{s^*} + \lambda^* & = & 0 \end{array} \tag{1.6}$$

HW 5 Problem

$$\begin{aligned} f(x) &= 0.25x_1^2 + x_s^2 \\ g(x) &= x_2 - 0.05x_1^2 - 0.5x_1 + 2 \end{aligned} \tag{3.18}$$

The gradient vectors are:

$$\begin{aligned} \nabla f(x) &= \begin{pmatrix} 0.5x_1 \\ 2x_2 \end{pmatrix} \\ \nabla g(x) &= \begin{pmatrix} -0.1x_1 - 0.5 \\ 1 \end{pmatrix} \end{aligned} \tag{3.19}$$

Kuhn-Tucker Solution

With 1.3 and 3.19, the necessary condition for the minimum is:

$$\begin{aligned} 0.5x_1^* + \mu^*(-0.1x_1^* - 0.5) &= 0 \\ 2x_2^* + \mu^* &= 0 \\ x_2^* - 0.05(x_1^*)^2 - 0.5x_1^* + 2 &\leq 0 \\ \mu^*(x_2^* - 0.05(x_1^*)^2 - 0.5x_1^* + 2) &= 0 \\ \mu^* &\geq 0 \end{aligned} \tag{3.20}$$

Like in Problem 1, $\mu = 0$ results in a violation of the inequality. The set of equations that satisfy the necessary condition for the minimum can therefore be written as:

$$\begin{aligned} 0.5x_1^* + \mu^*(-0.1x_1^* - 0.5) &= 0 \\ 2x_2^* + \mu^* &= 0 \\ x_2^* - 0.05(x_1^*)^2 - 0.5x_1^* + 2 &= 0 \end{aligned} \tag{3.21}$$

3.21 is a non-linear system of equations which can be solved with the Newton method. The solution is:

$$\begin{aligned} x_1^* &= 2.0896 \\ x_2^* &= -0.7368 \\ \mu^* &= 1.4737 \end{aligned} \tag{3.22}$$

Interior Point Method

Equations 3.19 and 3.18 in 1.6 result in a non-linear set of equations:

$$\begin{array}{rcrcrcl} h_1(x^*,\lambda^*,s^*) & = & \frac{\partial L}{\partial x_1}(x^*,\lambda^*,s^*) & = & 0.5x_1^* + \lambda^*(-0.1x_1^* - 0.5) & = & 0 \\ h_2(x^*,\lambda^*,s^*) & = & \frac{\partial L}{\partial x_2}(x^*,\lambda^*,s^*) & = & 2x_2^* + \lambda^* & = & 0 \\ h_3(x^*,\lambda^*,s^*) & = & \frac{\partial L}{\partial \lambda}(x^*,\lambda^*,s^*) & = & x_2^* - 0.05(x_1^*)^2 - 0.5x_1^* + 2 + s^* & = & 0 \\ h_4(x^*,\lambda^*,s^*) & = & \frac{\partial L}{\partial s}(x^*,\lambda^*,s^*) & = & -\frac{\mu^k}{s^*} + \lambda^* & = & 0 \end{array} \tag{3.23}$$

These can be solved with the Newton method.

H replaces J in Newton's Method

$$H = \begin{bmatrix} 0.5 - 0.1\lambda & 0 & -0.1x_1 & 0 \\ 0 & 2 & 1 & 0 \\ -0.1x_1 - 0.5 & 1 & 0 & 1 \\ 0 & 0 & 1 & \frac{\mu^k}{s^2} \end{bmatrix} \tag{3.24}$$

See following pages for Matlab program and sample results with different starting points.

```
function interior_point
%function to minimize the function f(x) = 0.25*x1^2 + x2^2

%           subject to constraint: g(x) = x2 - 0.05*x1^2 - 0.5*x1 + 2 <= 0
display(' ')

%initialize start values
x1     = 10;
x2     = 10;
lambda = 1;
s      = 2;
mu     = 20;
%mu     = 0.002;

eps    = 1E-4; %tolerance
alpha = 0.5;
display(' STARTING POINT ')
disp(['        mu         x1         x2         lambda        s          f(x)        g(x)']);
fofx   = 0.25*x1^2 + x2^2;
gofx   = x2 - 0.05*x1^2 - 0.5*x1 + 2;
fprintf(' %10.5f  %10.5f  %10.5f  %10.5f  %10.5f  %10.5f  %10.5f \n', mu, x1, x2, lambda, s, fofx, gofx );
display(' ')


x=[x1 x2 lambda s]';
display( 'ITERATIVE STEPS');
disp(['        mu         x1         x2         lambda        s          f(x)        g(x)']);
while 1;
    %mu loop, keep decreasing mu til x does not change anymore
    x_pre=x;
    while 1;
        %Newton loop, iterate till norm(h(x)) < eps
        H = Hessian(x, mu);
        rhs = h(x, mu);
        delta_x = -H \ rhs;
        if (norm(rhs)<eps)
            break;
        end
        x=x + alpha*delta_x;
    end
    x1     = x(1);
    x2     = x(2);
    lambda = x(3);
    s      = x(4);
    fofx   = 0.25*x1^2 + x2^2;
    gofx   = x2 - 0.05*x1^2 - 0.5*x1 + 2;
```

```
    fprintf(' %10.5f  %10.5f  %10.5f  %10.5f  %10.5f  %10.5f  %10.5f \n', mu, x1, x2, lambda, s, fofx, gofx );

    if (norm(x-x_pre)<eps)
        break;
    end

    mu=mu/2;
end;



function H = Hessian (x, mu)
%calculate the Hessian of the Newton solution

x1     = x(1);
x2     = x(2);
lambda = x(3);
s      = x(4);

H=[(0.5-0.1*lambda) 0 (-0.1*x1) 0;
    0               2     1     0;
   (-0.1*x1-.5)     1     0     1;
    0               0     1     mu/s^2 ];


function res = h(x, mu)
%calculate the function residuals which will become the new rhs
x1     = x(1);
x2     = x(2);
lambda = x(3);
s      = x(4);

res = zeros(4,1);

res(1) = 0.5*x1+lambda*(-0.1*x1-0.5);
res(2) = 2*x2 + lambda;
res(3) = x2 - 0.05*x1^2 - 0.5*x1 + 2 + s;
res(4) = -mu/s+lambda;
```

```
> HW4_interior_point

 STARTING POINT
        mu          x1          x2          lambda          s           f(x)          g(x)
   20.00000    10.00000    10.00000      1.00000      2.00000    125.00000      2.00000

ITERATIVE STEPS
        mu          x1          x2          lambda          s           f(x)          g(x)
   20.00000     7.85850    -1.52786      3.05577      6.54491     17.77336     -6.54491
   10.00000     5.66521    -1.32795      2.65591      3.76527      9.78711     -3.76529
    5.00000     4.24983    -1.14862      2.29723      2.17658      5.83459     -2.17659
    2.50000     3.35585    -1.00403      2.00806      1.24504      3.82351     -1.24504
    1.25000     2.80615    -0.89869      1.79738      0.69548      2.77626     -0.69549
    0.62500     2.48060    -0.82901      1.65801      0.37697      2.22559     -0.37697
    0.31250     2.29630    -0.78680      1.57360      0.19860      1.93731     -0.19860
    0.15625     2.19637    -0.76301      1.52602      0.10240      1.78819     -0.10240
    0.07813     2.14395    -0.75027      1.50053      0.05207      1.71203     -0.05207
    0.03906     2.11704    -0.74365      1.48730      0.02627      1.67348     -0.02627
    0.01953     2.10340    -0.74028      1.48056      0.01319      1.65408     -0.01319
    0.00977     2.09653    -0.73858      1.47715      0.00661      1.64435     -0.00661
    0.00488     2.09308    -0.73772      1.47544      0.00331      1.63948     -0.00331
    0.00244     2.09135    -0.73729      1.47458      0.00166      1.63704     -0.00166
    0.00122     2.09049    -0.73708      1.47415      0.00083      1.63582     -0.00083
    0.00061     2.09006    -0.73697      1.47394      0.00041      1.63521     -0.00041
    0.00031     2.08984    -0.73691      1.47383      0.00021      1.63490     -0.00021
    0.00015     2.08973    -0.73689      1.47377      0.00010      1.63475     -0.00010
    0.00008     2.08968    -0.73687      1.47375      0.00005      1.63467     -0.00005
```

Starting point near the final solution

```
> HW4_interior_point
```

STARTING POINT

mu	x1	x2	lambda	s	f(x)	g(x)
20.00000	2.00000	-0.73000	1.00000	2.00000	1.53290	0.07000

ITERATIVE STEPS

mu	x1	x2	lambda	s	f(x)	g(x)
20.00000	7.85841	-1.52788	3.05577	6.54482	17.77309	-6.54482
10.00000	5.66521	-1.32795	2.65591	3.76527	9.78711	-3.76529
5.00000	4.24983	-1.14862	2.29723	2.17658	5.83459	-2.17659
2.50000	3.35585	-1.00403	2.00806	1.24504	3.82351	-1.24504
1.25000	2.80615	-0.89869	1.79738	0.69548	2.77626	-0.69549
0.62500	2.48060	-0.82901	1.65801	0.37697	2.22559	-0.37697
0.31250	2.29630	-0.78680	1.57360	0.19860	1.93731	-0.19860
0.15625	2.19637	-0.76301	1.52602	0.10240	1.78819	-0.10240
0.07813	2.14395	-0.75027	1.50053	0.05207	1.71203	-0.05207
0.03906	2.11704	-0.74365	1.48730	0.02627	1.67348	-0.02627
0.01953	2.10340	-0.74028	1.48056	0.01319	1.65408	-0.01319
0.00977	2.09653	-0.73858	1.47715	0.00661	1.64435	-0.00661
0.00488	2.09308	-0.73772	1.47544	0.00331	1.63948	-0.00331
0.00244	2.09135	-0.73729	1.47458	0.00166	1.63704	-0.00166
0.00122	2.09049	-0.73708	1.47415	0.00083	1.63582	-0.00083
0.00061	2.09006	-0.73697	1.47394	0.00041	1.63521	-0.00041
0.00031	2.08984	-0.73691	1.47383	0.00021	1.63490	-0.00021
0.00015	2.08973	-0.73689	1.47377	0.00010	1.63475	-0.00010
0.00008	2.08968	-0.73687	1.47375	0.00005	1.63467	-0.00005

Starting point very far from solution

```
> HW4_interior_point
```

STARTING POINT

mu	x1	x2	lambda	s	f(x)	g(x)
20.00000	-10.00000	-10.00000	1.00000	2.00000	125.00000	-8.00000

ITERATIVE STEPS

mu	x1	x2	lambda	s	f(x)	g(x)
20.00000	-2.24509	2.03748	-4.07496	-4.90802	5.41143	4.90800
10.00000	-1.67781	1.26245	-2.52490	-3.96060	2.29754	3.96060
5.00000	-1.16871	0.76253	-1.52506	-3.27859	0.92292	3.27859
2.50000	-0.75835	0.44687	-0.89374	-2.79729	0.34347	2.79729
1.25000	-0.45934	0.25284	-0.50568	-2.47196	0.11668	2.47196
0.62500	-0.26162	0.13796	-0.27591	-2.26534	0.03614	2.26534
0.31250	-0.14189	0.07292	-0.14585	-2.14286	0.01035	2.14286
0.15625	-0.07440	0.03767	-0.07533	-2.07459	0.00280	2.07459
0.07813	-0.03825	0.01917	-0.03835	-2.03822	0.00073	2.03822
0.03906	-0.01947	0.00968	-0.01936	-2.01940	0.00019	2.01940
0.01953	-0.00989	0.00487	-0.00974	-2.00981	0.00005	2.00981
0.00977	-0.00504	0.00244	-0.00489	-2.00496	0.00001	2.00496
0.00488	-0.00259	0.00123	-0.00246	-2.00252	0.00000	2.00252
0.00244	-0.00136	0.00062	-0.00124	-2.00130	0.00000	2.00130
0.00122	-0.00081	0.00032	-0.00065	-2.00073	0.00000	2.00073
0.00061	-0.00050	0.00017	-0.00035	-2.00042	0.00000	2.00042
0.00031	-0.00034	0.00010	-0.00020	-2.00027	0.00000	2.00027
0.00015	-0.00027	0.00007	-0.00014	-2.00020	0.00000	2.00020
0.00008	-0.00020	0.00004	-0.00009	-2.00015	0.00000	2.00015
0.00004	-0.00020	0.00004	-0.00009	-2.00015	0.00000	2.00015

Here it does not converge, can you determine why?